Trammel System Force Analysis

1 Application of Newton's Second Law

Consider the application of the approach previously discussed to the determination of the forces in the trammel system shown in Figure 1.

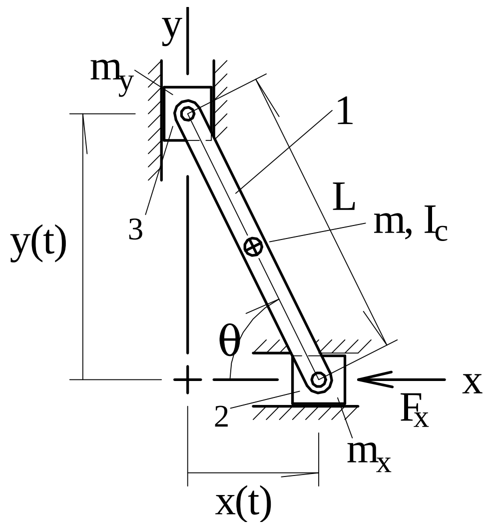

Figure 1: Example: Trammel System

The trammel consists of a single link with two sliders that move in guides along the horizontal $x-$ and $y-$ axes, respectively. Gravity is not a factor in this problem. The center of mass of the link is at the mid–point, so that $(x_c, y_c) = [x(\theta)/2, y(\theta)/2]$. The kinematics of this system are very simple indeed:

$$
\begin{aligned}
x &= L\cos\theta \\
y &= L\sin\theta \\
K_x &= -L\sin\theta \\
K_y &= L\cos\theta \\
L_x &= -L\cos\theta \\
L_y &= -L\sin\theta
\end{aligned}
$$

and the center of mass values are simply half of these values.

It is necessary to define the internal force components within the system, and for that purpose, consider Figure 2.

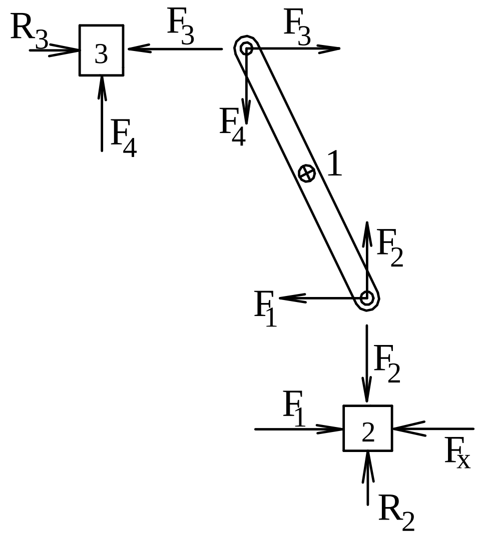

Figure 2: Forces Within the Trammel System

From Part 1, the general expressions for the Newton's Second Law equations for each body are of the form

$$
\begin{aligned}
\sum F_{xi} &= M_i \ddot{X}_{ci} = M_i \left[\ddot{q} K_{cxi}(q) + \dot{q}^2 L_{cxi}(q)\right] \\
\sum F_{yi} &= M_i \ddot{Y}_{ci} = M_i \left[\ddot{q} K_{cyi}(q) + \dot{q}^2 L_{cyi}(q)\right] \\
\sum^{+CCW} M_{ci} &= I_{ci} \ddot{\theta}_i = I_{ci} \left[\ddot{q} K_{\theta i}(q) + \dot{q}^2 L_{\theta i}(q)\right]
\end{aligned}
$$

Following the process outlined above, the equations of motion for the link, body 1, are:

$$
\begin{aligned}
-F_1 + F_3 &= m\left[\ddot{\theta}\left(-\frac{L}{2}\sin\theta\right) + \dot{\theta}^2\left(-\frac{L}{2}\cos\theta\right)\right] \\
F_2 - F_4 &= m\left[\ddot{\theta}\left(\frac{L}{2}\cos\theta\right) + \dot{\theta}^2\left(-\frac{L}{2}\sin\theta\right)\right] \\
\frac{L}{2}\left(F_1\sin\theta - F_2\cos\theta + F_3\sin\theta - F_4\cos\theta\right) &= I_c\left[\ddot{\theta}\cdot(1) + \dot{\theta}^2\cdot(0)\right]
\end{aligned}
$$

Note that positive moments on body 1 have been taken clockwise, rather than counter clockwise, in order to be consistent with the fact that the angle θ opens positive in the clockwise sense.

For the lower slider, body 2:

$$
\begin{aligned}
F_1 - F_x &= m_x \left[\ddot{\theta}\left(-L\sin\theta\right) + \dot{\theta}^2\left(-L\cos\theta\right)\right] \\
R_2 - F_2 &= m_x \left[\ddot{\theta}\cdot(0) + \dot{\theta}^2\cdot(0)\right] \\
0 &= I_{cx} \left[\ddot{\theta}\cdot(0) + \dot{\theta}^2\cdot(0)\right]
\end{aligned}
$$

Thus it is evident that, because of the one dimensional motion of the slider, there is only one actual equation of motion resulting. The other equations simply express static equilibrium.

For the other slider, body 3:

$$
F_4 = m_y \left[\ddot{\theta}\left(L\cos\theta\right) + \dot{\theta}^2\left(-L\sin\theta\right)\right]
$$

Collecting all of the nontrivial equations together gives the following group:

$$
\begin{bmatrix}
-1 & 0 & +1 & 0 & m\frac{L}{2}\sin\theta \\
0 & +1 & 0 & -1 & -m\frac{L}{2}\cos\theta \\
\frac{L}{2}\sin\theta & \frac{-L}{2}\cos\theta & \frac{L}{2}\sin\theta & \frac{-L}{2}\cos\theta & -I_c \\
1 & 0 & 0 & 0 & m_x L\sin\theta \\
0 & 0 & 0 & 1 & -m_y L\cos\theta
\end{bmatrix}
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \ddot{\theta} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ F_x \\ 0 \end{Bmatrix}
+\dot{\theta}^2
\begin{Bmatrix} -m\frac{L}{2}\cos\theta \\ -m\frac{L}{2}\sin\theta \\ 0 \\ -m_x L\cos\theta \\ -m_y L\sin\theta \end{Bmatrix}
$$

$$
[C]\{F\} = \{R_1\} + \dot{\theta}^2\{R_2\}
$$

2 System Equation of Motion

For the actually computing the dynamic response, it is much easier to use the equation of motion formulated using Eksergian's equation of motion as follows:

$$
\begin{aligned}
T &= \frac{1}{2}m\left(\dot{x}_c^2 + \dot{y}_c^2\right) + \frac{1}{2}I_c\dot{\theta}^2 + \frac{1}{2}m_x\dot{x}^2 + \frac{1}{2}m_y\dot{y}^2 \\
&= \frac{1}{2}\dot{\theta}^2\left[m\frac{L}{12}^2 + m_x\left(-L\sin\theta\right)^2 + m_y\left(-L\cos\theta\right)^2 + m\frac{L^2}{4}\left(\cos^2\theta + \sin^2\theta\right)\right] \\
&= \frac{1}{2}\dot{\theta}^2\left[m\frac{L}{3}^2 + m_x L^2\sin^2\theta + m_y L^2\cos^2\theta\right]
\end{aligned}
$$

from which the generalized inertia and centriptal coefficients are

$$
\begin{aligned}
\mathbb{I} &= m\frac{L}{3}^2 + m_x L^2 \sin^2\theta + m_y L^2 \cos^2\theta \\
\mathbb{C} &= \frac{1}{2}\frac{d\mathbb{I}}{d\theta} \\
&= (m_x - m_y)\, L^2 \sin\theta\cos\theta
\end{aligned}
$$

The virtual work of the external force is used to determine the generalized external force, Q_{ext}, as follows:

$$
\begin{aligned}
\delta W &= -F_x(t)\,\delta x = -F_x K_x \delta\theta \\
Q_{ext} &= -F_x(t)\, K_x
\end{aligned}
$$

Then the equation of motion is

$$
\begin{aligned}
\ddot{\theta}\mathbb{I} + \dot{\theta}^2\mathbb{C} &= Q_{ext} \\
\ddot{\theta} &= \frac{-F_x(t)\, K_x - \dot{\theta}^2 (m_x - m_y)\, L^2 \sin\theta\cos\theta}{m\frac{L}{3}^2 + m_x L^2 \sin^2\theta + m_y L^2 \cos^2\theta}
\end{aligned}
$$

This equation of motion is such that there is very little likelihood of solving it in closed form. This means that a numerical solution is definitely indicated.

3 Numerical Solution

There are many ways to solve this system numerically. The method outlined here is a particular version of the Fourth Order Runge–Kutta algorithm that is adapted specifically to second order systems. This is particularly useful in this case because it bypasses the need to break the differential equation into two first order differential equations, a step required in most numerical solution techniques. The algorithm follows the steps outlined below, beginning with four evaluations of the derivative for different arguments, and then an update of the solution estimate.

$$
\begin{aligned}
m_1 &= h\cdot f(t_n, q_n, \dot{q}_n) \\
m_2 &= h\cdot f\left(t_n + \frac{h}{2}, q_n + \frac{1}{2}\dot{q}_n h + \frac{1}{8}m_1 h, \dot{q}_n + \frac{1}{2}m_1\right) \\
m_3 &= h\cdot f\left(t_n + \frac{h}{2}, q_n + \frac{1}{2}\dot{q}_n h + \frac{1}{8}m_1 h, \dot{q}_n + \frac{1}{2}m_2\right) \\
m_4 &= h\cdot f\left(t_n + h, q_n + \dot{q}_n h + \frac{1}{2}m_3 h, \dot{q}_n + m_3\right)
\end{aligned}
$$

After these four derivative evaluations are made, each with their different arguments, then the solution estimates and the time are updated according to

$$
\begin{aligned}
q_{n+1} &= q_n + h\left[\dot{q}_n + (1/6)\left(m_1 + m_2 + m_3\right)\right] \\
\dot{q}_{n+1} &= \dot{q}_n + (1/6)\left(m_1 + 2m_2 + 2m_3 + m_4\right) \\
t_{n+1} &= t_n + h
\end{aligned}
$$

3.1 Observation

When the numerical solution for the equation of motion is made, one striking fact becomes evident if it has not been recognized previously: The equation of motion is highly nonstationary and nonlinear! Simply put, the solution is all over the place. This does not mean that there is anything wrong with the numerical process or with the modeling of the physical system as far as it has been taken, but rather it is in the nature of the differential equation of motion.

Intuition regarding the physical system suggests that is not quite right, so where is the difficulty? The problem lies in the fact that, to this point, no damping of any sort has been included in the model but any real system will have numerous energy loss mechanisms. One of the reasons they have not been included up to this point is that they are so very difficult to quantify, and secondly they are difficult to properly model. Even so, to obtain a more reasonable dynamic model, it is essential to include some form of damping.

Ideally the exact, physical damping should be modelled, but that is virtually unknowable for most physical systems. In this case, the simple expedient is taken to include a viscous damping force on the lower slider, body 2. On this basis then, the following modifications are introduced:

$$
\begin{aligned}
R_1(4) &= +b_x \dot{x}(t) = b_x K_x \dot{\theta} \\
Q_{ext}\delta\theta &= \left[-F_x(t) - b_x \dot{x}(t)\right]\delta x \\
&= \left[-F_x(t) - b_x K_x(\theta)\,\dot{\theta}\right] K_x(\theta)\,\delta\theta \\
Q_{ext} &= \left[-F_x(t) - b_x K_x(\theta)\,\dot{\theta} + c\right] K_x(\theta)
\end{aligned}
$$

This induces considerably more realism into the model results.

4 Numerical Example

Consider the specific case where the coupling link, body 1, is a steel bar, with $L = 30$ in., $w = 48.286875$ lb, $I_c = 9.3800264$ lb–s^2–in. For the $x-$axis slider, $w_x = 17$ lb, and for the $y-$axis slider, $w_y = 25$ lb. The force acting on the horizontal slider is $F_x(t)$

$$
\begin{aligned}
F_x(t) &= 50\sin(2\pi t) \qquad & 0 \le t \le 1 \\
&= 0 & t > 1
\end{aligned}
$$

The damping force is described by a viscous damping coefficient, $b_x = 0.5$

The system initial position is taken as $\theta(t=0) = 0.8$ rad $= 45.837^o$, just slightly above 45^o. The system is initially at rest, so that $\dot{\theta}(t=0) = 0$.

There is no gravity acting in the plane of the mechanism, so the internal forces are all zero to begin. With the inclusion of viscous damping and a time limited driving force, the system will eventually return to rest again (it may require an infinite time for this to happen). For this reason, the forces will eventually tend toward zero again. These are things that should be expected in the results.

The computed results are shown below. In Figure 3, System Dynamic Response, there are three curves shown. The sinusoid looping all the way to the top of the figure is the excitation function, $F_x(t)$, plotted as $F_x(t)/10$ so that it will fit on the scale; notice that this function goes to zero at $t = 1.0$. There is also a very smooth function that actually starts at $+0.8$, rises a bit, and then falls back toward that same level; this is the function $\theta(t)$ that describes the angular position of the trammel bar. The third curve is $\dot{\theta}(t)$ which is easily picked out because it starts at zero, rises and then goes to zero right below the maximum displacement. Notice that it is slowly approaching zero as the plot goes off the page to the right, although it is still not there. The final curve in the set is the angular acceleration of the bar, $\ddot{\theta}(t)$. The angular acceleration begins at zero, rises only to drop to zero at the maximum angular velocity point. It is a minimum slightly before the velocity goes to zero (this is just a coincidence), and is rising to a cusp at the instant when the force, $F_x(t)$ is abruptly ended. After the force ends, the angular acceleration sags away toward zero, rather than continuing to rise as it had before. This cusp is an important point in the motion and will be noted again.

Figure 4 is a plot of the forces on the lower pin joint, a plot of F_2 vertically against F_1 horizontally. Recall that when the motion begins, these forces are both zero, so the sequence of events can be traced from that point outward. Both forces are initially in the first quadrant, then pass into the fourth quadrant, then the third, and very briefly into the second quadrant before retracing the path rather closely back to the point where the force, $F_x(t)$ goes to zero. This is marked by the cusp on the curve in the first quadrant in the return leg, just as was seen in the acceleration curve. From that point on, both forces tend back toward the origin.

Figure 5 is the plot of F_4 versus F_3. As previously, both forces begin at the origin, and initially move into the first quadrant. Thereafter, the move to the second, third, fourth, and momentarily back into the first quadrant before retracing the path to a point in the second quadrant where there is a cusp. This is the point where the external driving force, $F_x(t)$ goes to zero. Thereafter, both forces tend back toward the origin.

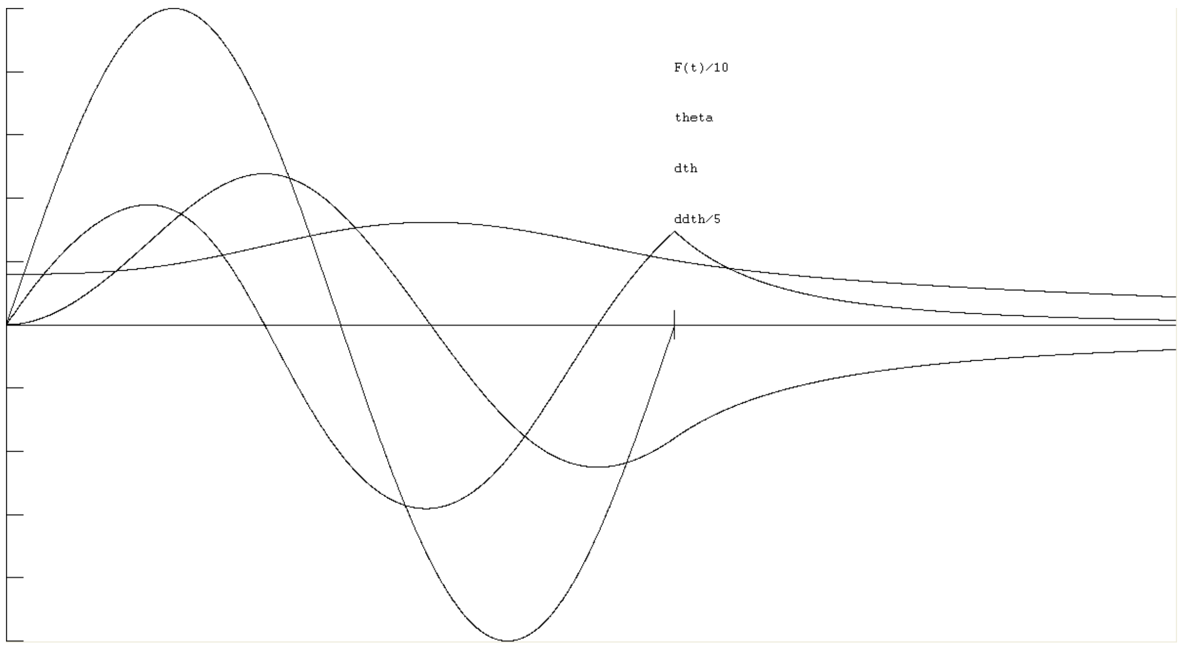

Figure 3: System Dynamic Response

5 Pitfalls

It will be fairly self–evident that there is many a slip twixt the cup and the lip in a problem such as this, all too many places where one can go off the rails in the calculations. On the other hand, with care, it should not be impossible to get a good solution, and indeed, it is not; only difficult!

As a guard against errors, it is important to build into any computer code checks to constantly test the validity of the results. Are you really getting the answers that you intended to get? There are two things that you can and should check constantly:

(1) Is the force solution a correct solution for the Newton's Second Law equations?

(2) Does the acceleration computed by Newton's Second Law agree with that computed by Eksergian's equation?

Both of these things must be true or you simply do not have a valid solution. To think otherwise is simply self deception.

It is very easy to have the computer substitute the numerical solution back into the Newton's Law equations, calculating a residual. If that residual is sufficiently small, the equation is taken to be satisfied. But how small is small, and how do you check each equation? First you do not check each equation, but rather you form the some of the squares of the residuals, something like

$$r^2 = r_1^2 + r_2^2 + ... + r_5^2$$

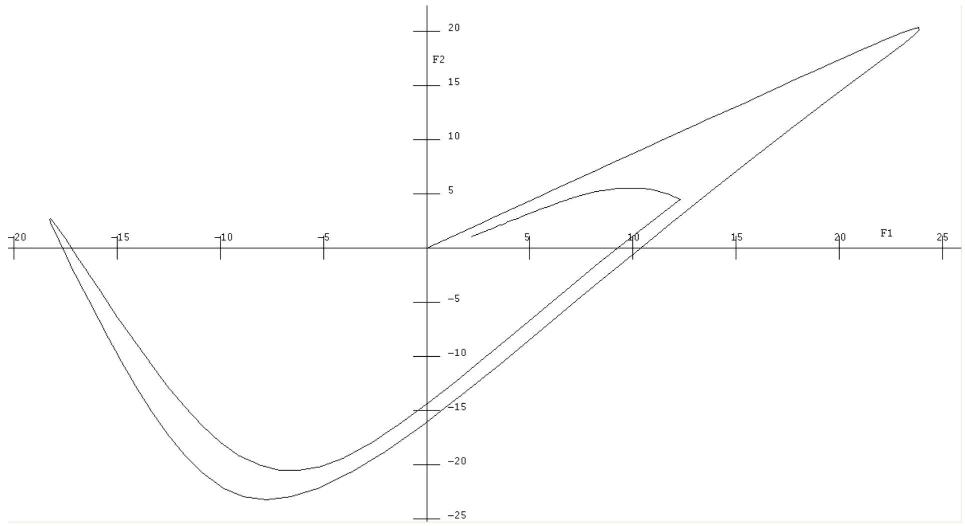

Figure 4: Polar Plot of F_2 versus F_1

This has the advantage that it gives you only a single number to be concerned with, rather than five, and it gets away from sign issues. You really do not care about the sign of the residual, only how large it is. For a good solution, typically, $|r_i| < 10^{-6}$ so that $r^2 \approx 10^{-12}$. If you test r^2 against 10^{-10}, then your results are probably good to at least five decimal places. You may do much better, depending on your computing system. But every force/acceleration solution should be checked.

By the same token, it is an easy thing to have the computer check the acceleration computed both through the solution of the Newton's Second Law equations and by Eksergian's equation. These must agree to at least seven or eight decimal places or you simply do not have a solution.

Automatic checking needs to be built into your computer code to always check these solutions to assure that you are getting the solutions you are setting out to get. Otherwise, the whole exercise is pointless!

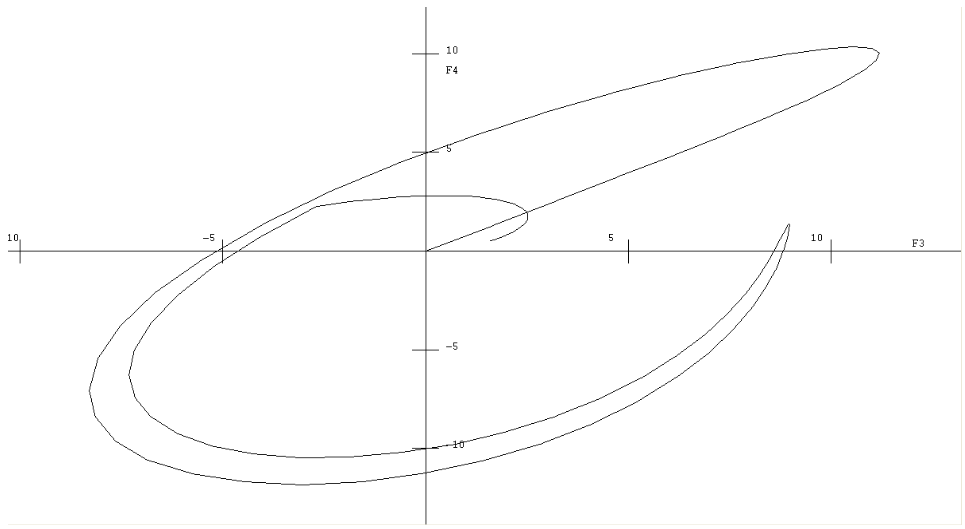

Figure 5: Polar Plot of F_4 versus F_3